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Free, transverse vibrations of thin plates with discontinuous boundary conditions

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Abstract

Vibrations of circular and rectangular plates clamped on part of the boundary and simply supported along the remainder are analyzed by means of a method of perturbation of boundary conditions. This approach appears to be simple and straightforward, giving excellent results for the first mode and its versatility permits to extend it to higher modes of vibration without difficulty. Furthermore, it is shown that the fundamental frequency coefficient can also be determined using a modified Galerkin approach and very simple polynomial coordinate functions which yield good engineering accuracy.

1. Introduction

The study of flexural vibrations of plates subjected to different boundary conditions has received considerable interest because of its great technological importance. Extensive and accurate data are available for classical boundary conditions: simply supported, clamped or free edges, as may be seen in Leissa's classical treatise [1].

It is a well-established fact that when the boundary conditions are uniform along an edge classical methods of analysis are applicable although, even for this situation, severe mathematical difficulties may arise (it is illuminating at this point to recall the problem of vibrations of a fully

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clamped rectangular plate which is probably one of the most difficult classical elastodynamics problems).

It is rather obvious that the degree of mathematical complexity increases drastically when one of the boundary conditions is not constant along a single boundary (Figs. 1–3).

If the two governing boundary conditions are not uniform the analytical approach may be not applicable, at least in some instances, and a considerably more sophisticated approach must be developed.

The problems we are dealing with belong to a general area defined as problems of mixed boundary conditions and one must resort to approximate methods for their solution.

Several authors have tackled these plate vibration problems using different, successful approaches.

Among them we must cite the important work of Bartlett [2] using a variational approach. He considered free transverse vibrations of a circular plate clamped on part of the boundary and simply supported on the remainder and calculated the fundamental frequency parameter giving upper and lower bounds for the eigenvalues.

It is also remarkable that a solution to this problem was given by Noble [3], who showed that a good approximation of the fundamental frequency parameter is given by the roots of a rather simple transcendental equation.

For the same problem but following a different approach, Narita and Leissa [4] obtained the natural frequencies and modes shapes for the fundamental and higher modes. This method was based on an extension of an analytical procedure developed previously by Leissa et al. [5]. It consisted of the consideration of a series-type solution and some uniform rotational springs along the circumference of the plate to reproduce the mixed boundary condition. A secular infinite determinant was obtained and the eigenvalues were determined as accurate as desired by successive truncation of it.

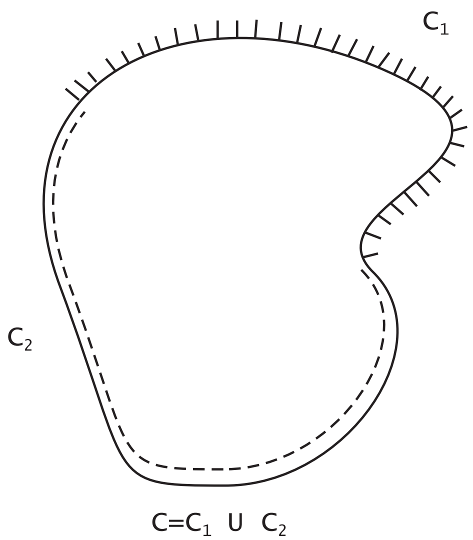

Fig. 1. Vibrating system with mixed boundary conditions: |||, clamped; – – –, simply supported.

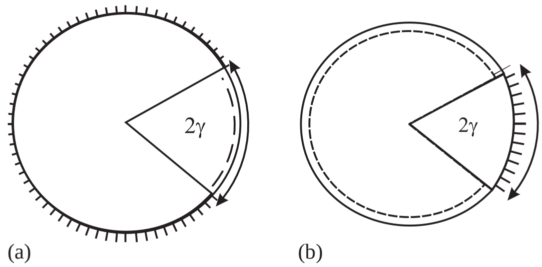

Fig. 2. Circular plates under study: (a) *quasi* fully clamped plate (b) *quasi* fully simply supported plate. | | |, clamped; − − −, simply supported.

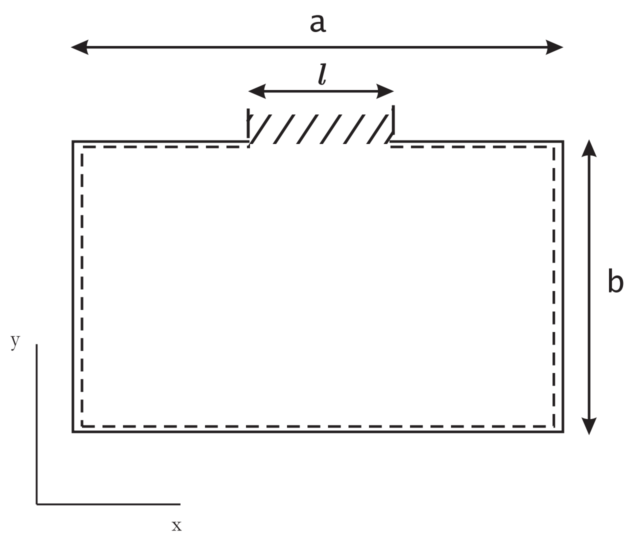

Fig. 3. Simply supported rectangular plate clamped along one symmetrically located segment of the edge $y = b$.

The problem was also ingeniously treated by Narita [6] from a different perspective following a method developed earlier by Irie et al. [7].

Rossi and Laura [8] used the finite element method to solve the problem and calculate the first nine frequencies of vibration of a circular plate with the following combination of boundary conditions: simply supported–clamped; simply supported–free and clamped–free.

The case of rectangular plates has also been treated when part of one edge is clamped while the remainder is simply supported.

Ota and Hamada [9] solved the problem by assuming a deflection function which satisfies the simply supported conditions everywhere and by applying distributed edge moments.

The problem was also solved by Kurata and Okamura [10] who used a very similar method. Both works show excellent agreement with the experimental results presented there.

In this study an extension of the method of perturbation of boundary conditions [11] to solve plate vibrations problems with mixed boundary conditions is presented.

The present work deals only with the situation where a portion of an edge is simply supported while the remainder is clamped. Since the displacement is null along the entire contour or edge C one has

$$W|_C = 0. \tag{1}$$

For the clamped portion C_1,

$$\left(\frac{\partial W}{\partial n}\right)\bigg|_{C_1} = 0, \tag{2}$$

where n is a coordinate normal to the contour.

For the simply supported segment C_2, the governing boundary condition is

$$(M_n)|_{C_2} = 0, \tag{3}$$

where M_n is the flexural moment normal to the contour defined as

$$M_n = -D\left(\frac{\partial^2 W}{\partial n^2} + \mu\kappa\,\frac{\partial W}{\partial n} + \mu\,\frac{\partial^2 W}{\partial s^2}\right),$$

and s refers to the coordinate tangential to the boundary surface, κ is the curvature of the boundary contour C, μ is Poisson's coefficient and D (flexural rigidity) is given by $Eh^3/12(1-\mu^2)$, where $E =$ Young's modulus, $h =$ plate thickness.

Summing up, this paper deals first with the implementation of the perturbation of boundary conditions approach. Then, calculations dealing with circular and rectangular plates, clamped on part of their boundaries and simply supported on the remainder, for the first mode of vibration are shown. After that, the procedure to obtain the eigenvalues of the problems for higher modes of vibration is sketched. Next, the calculations leading to the fundamental frequency for circular plates through a modified Galerkin method using very simple polynomial approximations are presented. Finally, the results are compared with those previously obtained.

2. Mathematical procedure

Considering a thin isotropic plate of uniform thickness h, mass density ρ, bounded by contour C and assuming simple harmonic motion at frequency ω, the equation of motion, which must satisfy the displacement amplitude, is

$$\nabla^4 W - k^4 W = 0. \tag{4}$$

If mixed boundary conditions are considered, the equations for W have to satisfy

$$W = 0 \text{ for } C, \quad \frac{\partial W}{\partial n} = 0 \quad \text{for } C_1, \tag{5,6}$$

$$\frac{\partial^2 W}{\partial n^2} + \mu\kappa\,\frac{\partial W}{\partial n} + \mu\,\frac{\partial^2 W}{\partial s^2} = 0 \text{ for } C_2, \tag{7}$$

where $k^4 = \rho h\omega^2/D$ is the eigenvalue under consideration.

In the procedure to follow it will be shown how the eigenvalue k is obtained from the eigenvalue for the unperturbed problem k^0, the solution for the unperturbed problem W^0 and the perturbed solution W. The method is applicable if the 'unperturbed' solution W^0 can be determined in a straightforward fashion and their eigenvalues k^0 are known exactly.

It is required that W be, in general, slightly different from the undisturbed solution W^0. Moreover, it must satisfy the plate's differential equation (4) and conveniently selected boundary conditions chosen in a way that they resemble as closely as possible the real problem. The next step is to obtain a single expression to calculate the actual eigenvalues k from the k^0's.

In order to develop an equation to calculate the eigenvalues k we shall make use of the well-known Green's second identity in two dimensions:

$$\int_S (u\nabla^2 v - v\nabla^2 u)\,\mathrm{d}A = \oint_C \left(u\,\frac{\partial v}{\partial n} - v\,\frac{\partial u}{\partial n}\right)\mathrm{d}l. \tag{8}$$

Taking $u = \nabla^2 W$ and $v = W^0$ in Eq. (8) results in the expression

$$\int_S (\nabla^2 W \nabla^2 W^0 - W^0 \nabla^4 W)\,\mathrm{d}A = \oint_C \left(\nabla^2 W\,\frac{\partial W^0}{\partial n} - W^0\,\frac{\partial \nabla^2 W}{\partial n}\right)\mathrm{d}l. \tag{9}$$

Now, substituting $u = W$ and $v = \nabla^2 W^0$ into Eq. (8) results in

$$\int_S (W \nabla^4 W^0 - \nabla^2 W^0 \nabla^2 W)\,\mathrm{d}A = \oint_C \left(W\,\frac{\partial \nabla^2 W^0}{\partial n} - \nabla^2 W^0\,\frac{\partial W}{\partial n}\right)\mathrm{d}l. \tag{10}$$

Since W and W^0 are assumed to satisfy the plate's differential equation (4) with eigenvalues k and k^0, respectively, adding Eqs. (9) and (10) leads to

$$(k^4 - (k^0)^4)\int_S W^0 W\,\mathrm{d}A = \oint_C \left(\nabla^2 W^0\,\frac{\partial W}{\partial n} - \nabla^2 W\,\frac{\partial W^0}{\partial n} + W^0\,\frac{\partial \nabla^2 W}{\partial n} - W\,\frac{\partial \nabla^2 W^0}{\partial n}\right)\mathrm{d}l. \tag{11}$$

Since one does not know the exact expression for W one must replace W by W^0 as it is normally done in a first-order perturbation procedure. This leads to the final result

$$(k^4 - (k^0)^4) = \frac{\oint_C \left(\nabla^2 W^0\,\frac{\partial W}{\partial n} - \nabla^2 W\,\frac{\partial W^0}{\partial n} + W^0\,\frac{\partial \nabla^2 W}{\partial n} - W\,\frac{\partial \nabla^2 W^0}{\partial n}\right)\mathrm{d}l}{\int_S (W_0)^2\,\mathrm{d}A}. \tag{12}$$

The above expression will be used as a starting point from which the eigenvalue k will be calculated for the different geometries and boundary conditions that will be emerging in this study.

To perform these calculations we will require some approximations selected specifically for each case under consideration.

The evaluation of Eq. (12) makes it possible to obtain the improved value k. Clearly one must choose a reasonably good approximation for the functional relation W since its exact expression is not known. To perform the required calculation for different geometries and boundary complications, it turns out to be necessary to assume convenient approximations for W and needed derivatives on the boundary. The procedure will be illustrated by two example cases.

3. Circular plates

In this section we apply the method to a circular plate partially clamped and partially simply supported on its contour.

As angle γ increases from 0 to π, the boundary of the plate changes from a *quasi* fully clamped plate with a very small segment which is simply supported (see Fig. 2(a)) to a *quasi* simply supported plate with a very small part which is clamped (see Fig. 2(b)). Because our method improves its precision as W^0 becomes similar to W, better results are obtained if this case is split into two cases.

As a first case, we consider $W(r,\theta)$ as the solution of the problem with the edge of the plate simply supported on an angle of $-\gamma<\theta<\gamma$, where $\gamma<\pi/2$, and clamped on the remainder. As $W^0(r,\theta)$ we will denote the solution of a circular plate totally clamped. A schematic representation of the plate is shown in Fig. 2(a).

In the second case we consider $W(r,\theta)$ as the solution of a circular plate where the boundary is clamped on $-\gamma<\theta<\gamma$ (where $\gamma<\pi/2$) and simply supported on the remainder (see Fig. 2(b)). On this occasion $W^0(r,\theta)$ will be the exact solution of a totally simply supported circular plate.

3.1. First case

As mentioned above, we choose W^0 as the exact solution of a fully clamped circular plate. Therefore it must satisfy the partial differential equation

$$\nabla^4 W^0(r,\theta) - (k^0)^4 W^0(r,\theta) = 0 \tag{13}$$

and the boundary conditions for a clamped plate:

$$W^0(r=a,\theta) = 0, \quad \left.\frac{\partial W^0}{\partial r}\right|_{r=a} = 0, \tag{14}$$

where a is the plate's radius.

For the perturbed problem we consider W as the displacement amplitude. Again it must satisfy

$$\nabla^4 W(r,\theta) - k^4 W(r,\theta) = 0. \tag{15}$$

The boundary conditions for the real problem are Eqs. (5)–(7). Here, for the perturbed problem, we replace them proposing

$$W(r=a,\theta) = 0 \tag{16}$$

and the approximate condition

$$\left.\frac{\partial W}{\partial r}\right|_{r=a} = c(\theta)\left.\frac{\partial W_{ss}}{\partial r}\right|_{r=a}, \tag{17}$$

where W_{ss} is the displacement amplitude for a simply supported circular plate and $c(\theta)$ is a step function which is 1 for $-\gamma<\theta<\gamma$ and 0 on the remainder.

These particular boundary conditions are proposed in a way that resembles as closely as possible the real boundary condition in each portion of the plate.

Then, to obtain the corrected eigenvalue of the problem k, we must solve Eq. (12).

Applying boundary conditions (14), (16) and (17) in Eq. (12), the resulting expression can be written as

$$(k^4 - (k^0)^4) = \frac{\oint_C \left(\nabla^2 W^0 c(\theta) \dfrac{\partial W_{ss}}{\partial r} \right) \mathrm{d}l}{\int_S (W^0)^2 \, \mathrm{d}A}. \tag{18}$$

Solving Eq. (18) the approximate eigenvalue k is obtained.

3.2. Second case

This time, we select W^0 as the exact solution of a totally simply supported circular plate. Then, it must satisfy the partial differential equation

$$\nabla^4 W^0(r,\theta) - (k^0)^4 W^0(r,\theta) = 0 \tag{19}$$

and boundary conditions for simply supported circular plates,

$$W^0(r=a,\theta) = 0, \quad \left. \frac{\partial^2 W^0}{\partial r^2} + \frac{\mu}{r} \frac{\partial W^0}{\partial r} \right|_{r=a} = 0. \tag{20}$$

For the perturbed problem W must satisfy

$$\nabla^4 W(r,\theta) - k^4 W(r,\theta) = 0, \tag{21}$$

with the perturbed boundary conditions. Now, we propose for this case

$$W(r=a,\theta) = 0, \quad \left. \frac{\partial^2 W}{\partial r^2} + \frac{\mu}{r} \frac{\partial W}{\partial r} \right|_{r=a} = c(\theta) \frac{\partial^2 W_c}{\partial r^2}, \tag{22}$$

where W_c represents the displacement amplitude for a totally clamped circular plate and $c(\theta)$ is a step function defined as 1 for $-\gamma < \theta < \gamma$ and 0 on the remainder.

From expression (12) and applying boundary conditions (20) and (22), one obtains

$$(k^4 - (k^0)^4) = \frac{\oint_C \left(\nabla^2 W^0 \dfrac{\partial W}{\partial r} - \nabla^2 W \dfrac{\partial W^0}{\partial r} \right) \mathrm{d}l}{\int_S (W^0)^2 \, \mathrm{d}A}. \tag{23}$$

The following criteria will be used for a first-order determination of Eq. (39). Since along the simply supported portion of the contour one has

$$\left. \frac{\partial^2 W^0}{\partial r^2} + \frac{\mu}{r} \frac{\partial W^0}{\partial r} \right|_{r=a} = 0, \tag{24}$$

one may approximate $\nabla^2 W_0$ by the boundary condition (24) since this will simplify considerably the calculation in view of the fact that in the line integral (39) it is multiplied by $\partial W/\partial r$ which is an unknown since W is not known.

Accordingly one now needs to evaluate the second term of the line integral. Here one again replaces $\nabla^2 W|_{r=a}$ by the nonhomogeneous boundary condition (22) which is a constitutive equation containing the effect of the clamped portion of the boundary,

$$\nabla^2 W|_{r=a} \cong \frac{\partial^2 W}{\partial r^2} + \frac{\mu}{r}\frac{\partial W}{\partial r}\bigg|_{r=a} = c(\theta)\,\frac{\partial^2 W_c}{\partial r^2}\bigg|_{r=a}. \tag{25}$$

Eq. (12) can then be rewritten to read

$$(k^4 - (k^0)^4) = -\frac{\oint_C c(\theta)\dfrac{\partial^2 W_c}{\partial r^2}\dfrac{\partial W^0}{\partial r}\,\mathrm{d}l}{\int_S (W^0)^2\,\mathrm{d}A}, \tag{26}$$

accordingly the desired, approximate eigenvalue k can be obtained from this equation.

4. Rectangular plates

In this instance we consider a rectangular plate in which the discontinuous boundary condition takes place on an edge where a particular coordinate is constant. For the implementation of the method we choose the edge $y = b$.

As in Section 3 the problem under consideration will be split into two cases to obtain more accurate results since the method improves its precision when W^0 approximates the real problem as closely as possible. The first case will be used when $a \geqslant l \geqslant a/2$ while the second one will be used when $0 \leqslant l < a/2$.

4.1. First case

On this occasion the solution for the unperturbed problem W^0 will be that of a rectangular plate simply supported on $x = 0,\ x = a,\ y = 0$ and clamped at $y = b$.

To determine the integral in Eq. (12) we need to evaluate W^0, W and their derivatives on the four sides of the rectangle.

The boundary conditions for W and W^0 establish that $W^0 = W = 0$ on all boundaries. Besides on $x = 0$ and $x = a$ one has

$$M_x^0 = -D\left(\frac{\partial^2 W^0(x,y)}{\partial x^2} + \mu\,\frac{\partial^2 W^0(x,y)}{\partial y^2}\right)\bigg|_{x=0,a} = 0, \tag{27}$$

$$M_x = -D\left(\frac{\partial^2 W(x,y)}{\partial x^2} + \mu\,\frac{\partial^2 W(x,y)}{\partial y^2}\right)\bigg|_{x=0,a} = 0, \tag{28}$$

while on $y = 0$,

$$M_y^0 = -D\left(\frac{\partial^2 W^0(x,y)}{\partial y^2} + \mu\,\frac{\partial^2 W^0(x,y)}{\partial x^2}\right)\bigg|_{y=0} = 0, \tag{29}$$

$$M_y = -D\left(\frac{\partial^2 W(x,y)}{\partial y^2} + \mu\,\frac{\partial^2 W(x,y)}{\partial x^2}\right)\bigg|_{y=0} = 0. \tag{30}$$

Finally, on the edge $y = b$ with the discontinuous boundary condition

$$\left.\frac{\partial W^0}{\partial y}\right|_{y=b} = 0 \tag{31}$$

and

$$\left.\frac{\partial W(x,y)}{\partial y}\right|_{y=b} = c(x)\left.\frac{\partial W_{ss}}{\partial y}\right|_{y=b}, \tag{32}$$

where W_{ss} is the displacement amplitude for a totally simply supported rectangular plate and $c(x)$ is a step function defined by

$$c(x) = \begin{cases} 1 & \text{for } 0 < x < a/2 - l/2, \\ 1 & \text{for } a/2 + l/2 < x < a, \\ 0 & \text{on the remainder.} \end{cases} \tag{33}$$

Eq. (32) establishes the boundary condition for the slope of W. We propose it to adopt that form for the same purpose as for the circular case.

The corrected eigenvalue for the problem will be obtained from the substitution of the equations for W and W^0 into Eq. (12). This leads to

$$(k^4 - (k^0)^4) = \frac{\oint_C \left(\nabla^2 W^0 \dfrac{\partial W}{\partial n} - \nabla^2 W \dfrac{\partial W^0}{\partial n}\right) \mathrm{d}l}{\int_S (W_0)^2 \, \mathrm{d}A}. \tag{34}$$

Applying the governing boundary conditions, Eqs. (27)–(32), we obtain

$$(k^4 - (k^0)^4) = \frac{\int_{y=b} \left(\nabla^2 W^0 \dfrac{\partial W}{\partial n}\right) \mathrm{d}l}{\int_S (W_0)^2 \, \mathrm{d}A}. \tag{35}$$

4.2. Second case

Consider now the same problem as in the first case. To obtain a better approximation to the real problem, it is convenient to choose W^0 as the solution of a simply supported rectangular plate over all sides. Therefore, W^0 will satisfy the same boundary conditions as in the previous case except for $y = b$ where one has

$$M_y^0|_{y=b} = -D\left(\frac{\partial^2 W^0(x,y)}{\partial y^2} + \mu \frac{\partial^2 W^0(x,y)}{\partial x^2}\right)\Bigg|_{y=b} = 0. \tag{36}$$

On the other hand, W will satisfy the same conditions as in the first case on $x = 0$, $x = a$ and $y = 0$ but now we model the discontinuous edge with the following constitutive condition:

$$M_y|_{y=b} = -D\left(\frac{\partial^2 W(x,y)}{\partial y^2} + \mu \frac{\partial^2 W(x,y)}{\partial x^2}\right)\Bigg|_{y=b} = -Dc(x) \left.\frac{\partial^2 W_{sc}(x,y)}{\partial y^2}\right|_{y=b}, \tag{37}$$

where W_{sc} is the displacement amplitude for a clamped plate on $y = b$ and simply supported on $x = 0$, $x = a$ and $y = 0$, and $c(x)$ is defined on the edge $y = b$, this time being equal to

$$c(x) = \begin{cases} 1 & \text{for } a/2 - l/2 < x < a/2 + l/2, \\ 0 & \text{on the remainder.} \end{cases} \tag{38}$$

Here the boundary conditions for W follow the same criteria as in Section 3.

The corrected eigenvalue of the problem, k, must be calculated from Eq. (12), once the consideration of null displacement of W_0 and W has been taken into account. Therefore, the final result is

$$(k^4 - (k^0)^4) = \frac{\oint_C \left(\nabla^2 W^0 \dfrac{\partial W}{\partial n} - \nabla^2 W \dfrac{\partial W^0}{\partial n} \right) \mathrm{d}l}{\int_S (W^0)^2 \, \mathrm{d}A}. \tag{39}$$

Knowing that W^0 is the solution of a totally simply supported rectangular plate we can set $\nabla^2 W^0 = 0$ on all sides of the rectangle.

On the other hand, we use the approximations

$$\nabla^2 W|_{x=0,a} \cong \frac{\partial^2 W(x,y)}{\partial x^2} + \mu \frac{\partial^2 W(x,y)}{\partial y^2}\bigg|_{x=0,a} = 0,$$

$$\nabla^2 W|_{y=0} \cong \frac{\partial^2 W(x,y)}{\partial y^2} + \mu \frac{\partial^2 W(x,y)}{\partial x^2}\bigg|_{y=0} = 0,$$

$$\nabla^2 W|_{y=b} \cong \frac{\partial^2 W(x,y)}{\partial y^2} + \mu \frac{\partial^2 W(x,y)}{\partial x^2}\bigg|_{y=b} = c(x) \frac{\partial^2 W_{sc}(x,y)}{\partial y^2}\bigg|_{y=b}.$$

Reaccommodating Eq. (39), we obtain

$$(k^4 - (k^0)^4) = -\frac{\int_{y=b} c(x) \dfrac{\partial^2 W_{sc}}{\partial n^2} \dfrac{\partial W^0}{\partial n} \, \mathrm{d}l}{\int_S (W^0)^2 \, \mathrm{d}A}. \tag{40}$$

Solving Eq. (40), k can be obtained.

5. Numerical results

5.1. Circular plate

To obtain the frequencies for the different modes, we must know the analytical solution of a totally simply supported and a totally clamped circular plate, which are

$$W_{ss} = (A_{nss} J_n(k_{nss} r) + C_{nss} I_n(k_{nss} r)) \begin{Bmatrix} \cos(n\theta) \\ \sin(n\theta) \end{Bmatrix}, \tag{41}$$

$$W_c = (A_{nc} J_n(k_{nc} r) + C_{nc} I_n(k_{nc} r)) \begin{Bmatrix} \cos(n\theta) \\ \sin(n\theta) \end{Bmatrix}, \tag{42}$$

where n denotes the number of nodal diameters and k_{nss} and k_{nc} correspond to the simply supported and clamped eigenvalues for the different modes, respectively.

5.1.1. Fundamental mode

For the case of $n = 0$, Eqs. (41) and (42) result in

$$W_{0ss} = A_{0ss}J_0(k_{0ss}r) + C_{0ss}I_0(k_{0ss}r), \tag{43}$$

$$W_{0c} = A_{0c}J_0(k_{0c}r) + C_{0c}I_0(k_{0c}r). \tag{44}$$

Substituting the above solutions into Eq. (18) for the first case (Section 3.1) and into Eq. (26) for the second case (Section 3.2) one obtains the desired value of the perturbed eigenvalue for the fundamental mode, k_0.

5.1.2. Higher modes ($n>0$)

Considering higher modes of vibration, it is fairly known that, for a circular plate with uniform boundary conditions ($\gamma = 0$), a two-fold degeneracy of the vibrational modes appears. If $\gamma \neq 0$ (as in our case), which means a discontinuous boundary condition, this degeneracy is splitting each mode into two, distinguishable one from the other. We will label these modes 'symmetric' and 'antisymmetric'. It is illustrative to point out that obviously no pure symmetric or antisymmetric modes exist in view of the geometric arrangement of the boundary conditions.

We restrict our analysis to the case $n = 1$. For this occasion the analytical solutions are obtained from Eqs. (41) and (42) substituting $n = 1$. Clearly, the expressions containing $\cos(\theta)$ correspond to the 'symmetrical' solution whereas the one with the $\sin(\theta)$ term to the 'antisymmetrical' one.

5.2. Rectangular case

This time we will consider the fundamental mode to illustrate the procedure to follow for rectangular plates. The extension of the method for higher modes is straightforward and implies the same considerations as in the circular plate.

The solution for the fundamental mode has been extracted from Ref. [1]. For a simply supported rectangular plate on the four edges, the exact analytical solution is

$$W_{ss} = A_{ss}\left(\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)\right). \tag{45}$$

If the rectangular plate is clamped on $y = b$ and simply supported on $x = 0,\ x = a$ and $y = 0$ the solution is

$$W_{sc} = A_{sc}\sin(\alpha x)(\sin(\lambda_1 b)\sinh(\lambda_2 y) - \sinh(\lambda_2 b)\sin(\lambda_1 y))\sin(\alpha y/b)), \tag{46}$$

where

$$\lambda_1 = \sqrt{k_{sc}^2 - \alpha^2}, \quad \lambda_2 = \sqrt{k_{sc}^2 + \alpha^2}, \quad \alpha = \pi/a,$$

where k_{sc} is the corresponding eigenvalue. These solutions will be used in Eqs. (35) and (40) to obtain the numerical values of the eigenfrequencies of the problem.

5.3. Fundamental frequency by a modified Galerkin method

In this section, we develop a 'modified Galerkin' method to obtain the fundamental frequency for circular plates clamped on $-\gamma<\theta<\gamma$ and simply supported on the remainder (see Fig. 2(b)). Evidently, the same procedure can be extended to analyze higher modes of vibration.

We resume the procedure as follows. First, the fundamental mode is approximated by a simple polynomial which identically satisfies the boundary conditions on each portion of the plate's edge, which are on $r=a$:

$$W=0, \quad \text{for } 0<\theta<2\pi; \tag{47}$$

$$\frac{\partial W}{\partial r}=0, \quad \text{for } -\gamma<\theta<\gamma; \tag{48}$$

$$\frac{\partial^2 W}{\partial r^2}+\frac{\mu}{r}\frac{\partial W}{\partial r}+\frac{\mu}{r^2}\frac{\partial^2 W}{\partial \theta^2}=0, \quad \text{for } \gamma\leqslant\theta<2\pi-\gamma. \tag{49}$$

In view of the 'quasi' radial symmetry of the mechanical system under study for the fundamental mode, one can approximate W by a polynomial term of the form [12]

$$W_a=A_1(\alpha r^4+\beta r^2+1), \tag{50}$$

where the subscript a stands for the approximate displacement function of the real problem W, and A_1, α and β are constants to be determined.

Considering Eqs. (48) and (49), they may be condensed into a single form as follows:

$$\left.\frac{\mathrm{d}W_a}{\mathrm{d}r}\right|_{r=a}=-\phi(\theta)D\left.\left(\frac{\mathrm{d}^2 W_a}{\mathrm{d}r^2}+\frac{\mu}{r}\frac{\mathrm{d}W_a}{\mathrm{d}r}\right)\right|_{r=a}, \tag{51}$$

where $\phi(\theta)$ is a step function defined by

$$\phi(\theta)=\begin{cases}0 & \text{for } -\gamma<\theta<\gamma,\\ \infty & \text{for } \gamma\leqslant\theta<2\pi-\gamma,\end{cases} \tag{52}$$

and where ∞ is replaced by 10^7–10^8 for numerical calculation purposes.

To determine the values of α and β, Eq. (50) must be substituted into the boundary condition (51). Then,

$$\alpha=\frac{(\phi D/a)(1+\mu)+1}{a^4(\phi D/a(5+\mu)+1)}, \quad \beta=\frac{-2}{a^2}\left[\frac{(\phi D/a)(3+\mu)+1}{(\phi D/a(5+\mu)+1)}\right]. \tag{53}$$

Accordingly, applying Eq. (53) and the definition of $\phi(\theta)$ given by Eq. (52), W_a results:

$$W_a=\begin{cases}A_{1c}(\alpha_c r^4+\beta_c r^2+1) & \text{for } -\gamma<\theta<\gamma,\\ A_{1s}(\alpha_s r^4+\beta_s r^2+1) & \text{for } \gamma\leqslant\theta<2\pi-\gamma,\end{cases} \tag{54}$$

where

$$\alpha_c=\alpha(\phi=0), \quad \beta_c=\beta(\phi=0), \quad \alpha_s=\alpha(\phi\to\infty), \quad \beta_s=\beta(\phi\to\infty).$$

We denote $(\alpha_{c(s)}r^4+\beta_{c(s)}r^2+1)$ as the coordinate function.

Substituting Eq. (54) into plate's governing differential equation (4), one obtains an 'error' function; $\varepsilon_c(r, A_{1c}, k(\omega))$ and $\varepsilon_s(r, A_{1s}, k(\omega))$ for the clamped and simply supported section of the plate. In this modified Galerkin method it is required that the sum of the error functions $\varepsilon_{c(s)}$ be orthogonal with respect to each coordinate function over the domain under consideration.

This condition leads to

$$\int_{-\gamma}^{\gamma}\int_0^a \varepsilon_c(r,\omega)(\alpha_c r^4+\beta_c r^2+1)r\,\mathrm{d}r\,\mathrm{d}\theta+\int_{\gamma}^{2\pi-\gamma}\int_0^a \varepsilon_s(r,\omega)(\alpha_s r^4+\beta_s r^2+1)r\,\mathrm{d}r\,\mathrm{d}\theta=0, \tag{55}$$

where $\varepsilon_{c(s)}(r,\omega)$ results from

$$D\nabla^4 W_a(r)-\rho h\omega^2 W_a(r)=\varepsilon_{c(s)}(r,\omega).$$

The root of expression (55), ω_{00}, is the fundamental approximate natural frequency of the problem.

The expression for the fundamental frequency coefficient $\Omega_{00}=\sqrt{(\rho h/D)}\,\omega_{00}a^2$ becomes

$$\Omega_{00}=64\left[\frac{\gamma\eta_c+(\pi-\gamma)\eta_s}{\gamma\delta_c+(\pi-\gamma)\delta_s}\right], \tag{56}$$

where

$$\eta_{c(s)}=\alpha_{c(s)}\left(\alpha_{c(s)}\frac{a^6}{6}+\beta_{c(s)}\frac{a^4}{4}+\frac{a^2}{2}\right),$$

$$\delta_{c(s)}=\alpha_{c(s)}^2\frac{a^{10}}{10}+\alpha_{c(s)}\beta_{c(s)}\frac{a^8}{8}+\beta_{c(s)}^2\frac{a^6}{6}+\alpha_{c(s)}\frac{a^6}{3}+\beta_{c(s)}\frac{a^4}{2}+\frac{a^2}{2}.$$

The original Galerkin method yields upper bounds. However, the results obtained by this 'modified Galerkin' approach are not necessarily upper bounds.

5.4. Discussion and results

Table 1 shows the fundamental frequency coefficient $\Omega_{00}=k_0^2a^2=\sqrt{(\rho h/D)}\,\omega_{00}a^2$ for the first or fundamental mode of vibration of a circular plate for different values of γ which here represents the clamped portion of the boundary as in Fig. 2(b).

It is important to point out that the values of comparison contained in Refs. [2,8] have been obtained for $\mu=0.25$, while the eigenvalues from Ref. [4] were calculated for $\mu=0.3$. Obviously this fact influences the values of Ω_{00}, with the exception of the case corresponding to the fully clamped plate ($\gamma=\pi$). However, it can be noticed that using $\mu=0.25$ or $\mu=0.3$ does not influence greatly the eigenvalues with the only exception of the cases $\gamma=0$ and $3/4\pi$, where the difference is of the order of 2%. Certainly for $\gamma=\pi$ (fully clamped case), Poisson's ratio does not come into play and the agreement is excellent. For $\gamma=0$ (fully simply supported plate) the present approach, the boundary perturbation method (BPM), yields the exact frequency coefficient in view of the inherent formulation of the proposed approach.

Table 1
Frequency coefficient Ω_{00} for the *fundamental* mode for a circular plate clamped on $-\gamma<\theta<\gamma$ and simply supported on the remainder; $\mu=0.25$ ($^*\mu=0.3$)

	γ								
	0	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$	$5\pi/8$	$3\pi/4$	$7\pi/8$	π
Ref. [2]	4.860	5.871	6.350	6.880	7.508	8.231	9.120	9.885	10.216
		5.842	6.335	6.864	7.480	8.162	8.880	9.126	
Ref. [4]*	4.937		6.487		7.673		9.284		10.216
Ref. [8]	4.868	5.859	6.351	6.878	7.511	8.272	9.118	9.88	10.216
BPM	4.860	5.6806	6.397	7.040	7.630	8.415	9.055	9.653	10.216
Mod. Galerkin	4.872	5.572	6.245	6.905	7.563	8.228	8.905	9.604	10.328

Note: BPM: Boundary perturbation method.

Table 2
Frequency coefficients for the first '*symmetric*' Ω_{1s} and '*antisymmetric*' Ω_{1a} modes for a circular plate clamped on $-\gamma<\theta<\gamma$ and simply supported on the remainder; $\mu=0.25$ ($^*\mu=0.3$)

	Mode type	γ								
		0	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$	$5\pi/8$	$3\pi/4$	$7\pi/8$	π
Ref. [8]	S	13.842	15.939	16.904	17.404	17.485	17.590	18.276	19.980	21.262
	A	13.842	14.070	14.748	15.886	17.550	19.493	20.833	21.232	21.262
Ref. [4]*	S	13.898		17.065		17.506		18.533		21.252
	A	13.898		14.946		17.910		20.960		21.252
BPM	S	13.835	15.690	16.902	17.429	17.511	17.803	18.403	19.652	21.260
	A	13.835	13.938	14.572	15.871	17.511	19.475	20.660	21.180	21.260

This is also the case for the fully clamped plate ($\gamma=\pi$). The modified Galerkin scheme is also presented and shows good agreement with referenced values.

Table 2 illustrates numerical values for the frequency coefficient for the first 'symmetric' ($n=1$) Ω_{1s} and first 'antisymmetric' Ω_{1a} modes ($\mu=0.25$). BPM calculations are presented for these cases which show very good agreement with referenced values.

Evidently, to obtain the results with BPM for $\gamma>\pi/2$, the formulation of the *first case* (Fig. 2(a)) presented in Section 3 was used whereas for values of $\gamma\leqslant\pi/2$ the *second case* (Fig. 2(b)) was considered.

In Table 3, the fundamental frequency coefficient for a simply supported square plate clamped along one symmetrically located segment of the edge $y=b$ is presented. The clamped portion of the edge is represented by l/a, where $l/a=1$ means a totally clamped edge. Exact values are available for the limiting cases of l/a while Ref. [9] was used to compare with the BPM (present) approach. A reasonable agreement between them is shown to exist.

In a similar fashion as for circular plates, the *first case* of Section 4 was used for plates in the case where $l/a>1/2$, whereas the *second case* was used when $l/a\leqslant 1/2$.

Table 3
Fundamental frequency coefficient for a square plate clamped along one symmetrically located segment of an edge; $\mu = 0.3$

	l/a						
	0	1/6	1/3	1/2	2/3	5/6	1
BPM	19.74	21.070	22.162	23.001	23.446	23.623	23.65
Ref. [9]	19.74		23.0	23.4	23.6		23.65

6. Concluding remarks

In the first place, the present novel methodology of solving this type of problem of discontinuous boundary conditions results in a simpler and less mathematical effort than other approaches by previous authors. For example, the finite element formulation for this problem in the case of a circular plate [8] has to use a high density mesh to solve the problem, in order to obtain a similar precision as in the present BPM approach. Secondly, the method can be extended straightforwardly to calculate higher modes of vibration without any substantial modification of the mathematical procedure.

It was demonstrated that excellent agreement is found for the first two modes of vibration for circular plates.

In the case of rectangular plates the method presents significantly good accuracy, considering the complexity of the problem and the simplicity of the analysis.

Another remarkable point, as may be deduced from the present analysis, is that this method can be applied to wide variations of plate shapes. Admittedly, the required algorithm may turn out to be quite complex, however.

Certainly a second-order perturbation, using the first-order perturbation mode shape, will improve the accuracy of the results but the complexity of the algorithmic procedure will increase considerably.

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